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HOROSPHERIC FOLIATIONS AND RELATIVE PINCHING

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Abstract

Relative curvature pinching in negative curvature provides regularity of the horospheric foliations up to $C^{2-\epsilon}$.

The horospheric foliations of a negatively curved Riemannian manifold are defined as the stable and unstable foliations of its geodesic flow, as explained below. There are two classical results about smoothness of horospheric foliations: Negatively curved surfaces have C^1 horospheric foliations [4], and $\frac{1}{4}$-pinched Riemannian manifolds have C^1 horospheric foliations [2]. The latter has been improved to give $C^{2\sqrt{a}}$ foliations assuming a-pinching ($a \in (0, 1)$). An open question, posed in [2], is whether these results hold assuming only relative pinching (e.g., does relative $\frac{1}{4}$-pinching imply C^1 foliations). We do not know the answer, but give sufficient relative pinching conditions for the same range of smoothness and indicate where improvements seem possible. See [1] for a brief survey of interesting related results.

Definition 1. The sectional curvature of a compact negatively curved Riemannian manifold N is *relatively a-pinched* if $C \leq$ sectional curvature $< aC$ for some $C: N \to -\mathbb{R}_+$. If C is constant, the curvature is said to be (absolutely) a-pinched.

Theorem 2. *For $a \in (0, 1)$ a compact relatively a-pinched Riemannian manifold has C^{2a} horospheric foliations.*

This follows from Theorems 5 and 6. Theorem 5 is a regularity theorem for the stable and unstable foliations of an Anosov flow based on a "bunching" assumption of contraction and expansion rates sharpening the standard regularity theory in [1], which cannot be substantially improved. Theorem 6 establishes a connection between relative pinching and bunching which may not be optimal. Here are the needed properties

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of the geodesic flow of a negatively curved Riemannian manifold.

Definition 3. A flow φ^t on a compact Riemannian manifold M is called Anosov with Anosov splitting $(E^u, E^s) := (E^{su} \oplus E^\varphi, E^{ss} \oplus E^\varphi)$ if $TM = E^{su} \oplus E^{ss} \oplus E^\varphi$, $E^\varphi = \text{span}\{\dot\varphi\} \neq \{0\}$, and $\exists \lambda < 1, C > 0$, $\forall p \in M$, $t > 0$ such that

$$\|D\varphi^t(v)\| \leq C\lambda^t \|v\| \qquad (v \in E^s(p))$$

and

$$\|D\varphi^{-t}(u)\| \leq C\lambda^{-t}\|u\| \qquad (u \in E^u(p)).$$

Call φ^t α-bunched if there exist $\mu_f \leq \mu_s < 1 < \nu_s \leq \nu_f : M \times \mathbb{R}_+ \to \mathbb{R}_+$ with

$$\lim_{t\to\infty} \sup_{p\in M} \mu_s(p, t)\nu_s(p, t)^{-1}\mu_f(p, t)^{-\alpha} = 0,$$

$$\lim_{t\to\infty} \sup_{p\in M} \mu_s(p, t)\nu_s(p, t)^{-1}\nu_f(p, t)^{\alpha} = 0$$

such that for all $p \in M$, $v \in E^{ss}(p)$, $u \in E^{su}(\varphi^t p)$, $t > 0$, we have

$$\mu_f(p, t)\|v\| \leq \|D\varphi^t(v)\| \leq \mu_s(p, t)\|v\|,$$

$$\nu_f(p, t)^{-1}\|u\| \leq \|D\varphi^{-t}(u)\| \leq \nu_s(p, t)^{-1}\|u\|.$$

This notion of bunching is weaker than the one used in [1]. For geodesic flows in negative curvature the terminology is clearer since $\mu_i = \nu_i^{-1}$ ($i = f, s$) by symplecticity, and hence α-bunching means

$$\lim_{t\to\infty} \sup_{p\in M} \nu_s(p, t)^{-2/\alpha}\nu_f(p, t) = 0,$$

so $\nu_s \leq \nu_f < \nu_s^{2/\alpha}$ uniformly for large t.

E^u and E^s are tangent to foliations W^u and W^s, respectively (unstable/stable foliations), whose leaves are C^∞ injectively immersed cells depending continuously on the base point in the C^∞ topology [3]. In the case of a geodesic flow these are the horospheric foliations on the unit tangent bundle. The regularity of E^u, E^s in the C^∞-topology is that of their representations in smooth local coordinates. Regularity of horospheric foliations is the regularity of their tangent distributions. For regularity C^1 and higher this coincides with all alternative definitions.

Definition 4. A map f between metric spaces is called Hölder continuous with exponent $\alpha \in (0, 1]$ if $d(f(x), f(y)) \leq \text{const} \cdot (d(x, y))^\alpha$ for nearby x and y. If $\beta \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$, f is $C^{[\beta]}$ and $f^{([\beta])}$ is $\beta - [\beta]$-Hölder, then we say $f \in C^\beta$. A distribution is C^β if it is C^β in a smooth chart.

Theorem 5. *The Anosov splitting of an α-bunched Anosov flow is C^α in the C^∞-topology for $\alpha \in (0, 2)$.*

Theorem 6. *A geodesic flow on the unit tangent bundle $M = SN$ of a compact relatively a-pinched Riemannian manifold N is $2a + \epsilon$-bunched for some ϵ.*

Remark. a-pinching implies $2\sqrt{a} + \epsilon$-bunching [6, Theorem 3.2.17], which is stronger. Ideally this would follow already from relative a-pinching.

Proof of Theorem 5. We only treat the case $\alpha \in (0, 1)$ to show how to modify [1]. The framework of the argument is the same as in [1] which in turn uses the formulation of [5]. For $p \in M$, take a hypersurface $\mathcal{T}_p$ transversal to $\dot{\varphi}$ of uniform size depending C^∞ on p. For each p, let $W^u := W^u(p) \cap \mathcal{T}_p$, $W^s := W^s(p) \cap \mathcal{T}_p$, $E^u := TW^u$, and $E^s := TW^s$. Take coordinates $\Xi: M \times [-\epsilon, \epsilon]^{k+l} \to M$ such that $\Xi_p: [-\epsilon, \epsilon]^{k+l} \xrightarrow{C^\infty} \mathcal{T}_p$ is continuous in p, $[-\epsilon, \epsilon]^k \times \{0\} \to W^u$, $\{0\} \times [-\epsilon, \epsilon]^l \to W^s$, and if $\phi^t: \mathcal{T}_p \to \mathcal{T}_{\varphi^t p}$ is the induced map then

$$D\phi^t\big|_0 = \begin{pmatrix} A_t & 0 \\ 0 & C_t \end{pmatrix}$$

with $\|A_t^{-1}\| < \nu_s(p, t)^{-1}$ and $\|C_t\| < \mu_s(p, t)$. Write the coordinates as (x, y) with $\Xi_p(x, 0) \in W^u$ and $\Xi_p(0, y) \in W^s$.

Lemma 7. *Given $p \in M$, $q \sim (0, y) \in W^s$, $(0, y_t) := \phi^t(0, y)$, there exist $C > 0$ and $C_t > 0$ such that*

$$D\phi^t = \begin{pmatrix} A_t & 0 \\ B_t & C_t \end{pmatrix}$$

with $\|A_t^{-1}\| < C\nu_s(q, t)^{-1}$, $\|C_t\| < C\mu_s(q, t)$, $\|C_t^{-1}\| < C\mu_f(q, t)^{-1}$, $\|B_t\| < C_t\|y\|$, $C\|y_t\| \geq \mu_f(q, t)\|y\|$.

Proof. $\|A_t^{-1}\| < \nu_s(q, t)^{-1}$ in coordinates centered at q. But up to a distortion factor, uniformly bounded independently of t, the linear part of the coordinate change is of the form $\begin{pmatrix} I & O \\ D & I \end{pmatrix}$, so that up to a bounded factor the representations A_t^{-1} agree in both systems, as do the ones for C_t and C_t^{-1}. $\|B_t\| < C_t\|y\|$ since φ^t is a diffeomorphism with B_t differentiable and vanishing at the origin of the coordinate system. For the remaining claim it is slightly easier and by boundedness of coordinate changes clearly sufficient to show $\|y\| \leq C\mu_f(p, t)^{-1}\|\phi^t(y)\|$. To this end let $\gamma_t: [0, 1] \to \mathcal{T}_{\varphi^t p}$ be a geodesic with $\gamma_t(0) = \phi^t(p)$, $\gamma_t(1) = \phi^t(q)$, where $q \sim (0, y)$. By standard hyperbolic theory $\phi^{-t}\gamma_t$ converges to a smooth curve $c(\cdot) \subset \mathcal{T}_p$. If $\lim_{n\to\infty} \|\phi^t(y)\|/\mu_f(p, t)\|y\| = 0$, then by the intermediate value theorem this holds for all $c(s)$, $s \in (0, 1]$. Using compactness of M (to control higher derivatives) yields uniformity in s, so $\lim_{n\to\infty} \|D\phi^t(v)\|/\mu_f(p, t)\|v\| = 0$ for $v = \dot{c}(0)$, contrary to the choice of μ_f. q.e.d.

In Ξ_p, represent elements

$$v \in V(\delta) := \left\{ \begin{array}{l} k+1\text{-dimensional distributions } v \text{ on } M \text{ such} \\ \text{that } v(p) \text{ contains } \dot{\varphi}(p) \text{ and is } \delta\text{-close to } E^u(p) \end{array} \right\}$$

by identifying $v(p)$ with $v(p) \cap T\mathcal{T}_p$; likewise for $v(q)$ in coordinates Ξ_p for $q \in \mathcal{T}_p$. Thus δ-closeness is determined by representing $v(p)$ as the graph of a linear map $D: \mathbb{R}^k \to \mathbb{R}^l$ via Ξ_p and using the norm topology. φ^t acts on $V(\delta)$ via $(\mathcal{P}_t v)(p) := D\varphi^t(v(\varphi^{-t}p))$. $\mathcal{P}_t(V(\delta)) \subset V(\delta)$ for large t and $\mathcal{P}_t v \overset{t\to\infty}{\longrightarrow} E^u$ for $v \in V(\delta)$. Also one easily shows

Lemma 8. *For $\delta, \epsilon_0 > 0$ there exists $K = K(\delta, \epsilon_0) > 0$ such that $V(\delta) \subset V(\delta, \epsilon_0, K) := \{E \in V(\delta) \,|\, \|E(z)\| \le K\|z\|^\alpha$ when $\epsilon_0 \le z \le \epsilon\} \subset V(\delta)$.*

This is useful since for all sufficiently large t we have $\mathcal{P}_t(V(\delta)) \subset V(\delta)$.

Proposition 9. *If $\alpha \in (0, 1]$ and φ^t is α-bunched, then E^u is C^α.*

This follows from

Lemma 10. *For all $\delta \in (0, 1)$ there exist $K > 0$, $\eta \in (0, 1)$ such that for all sufficiently large t we have $\mathcal{P}_t(V(\delta)) \subset V(\delta, \eta^t, K)$.*

Namely $\bigcap_{i\in\mathbb{N}} \mathcal{P}_t(V(\delta)) \subset V(\delta, 0, K)$, i.e., every $E \in \bigcap_{i\in\mathbb{N}} \mathcal{P}_t(V(\delta))$ is Hölder continuous with exponent α and constant K. But by construction we have $E^u \in \bigcap_{i\in\mathbb{N}} \mathcal{P}_t(V(\delta))$.

To obtain Lemma 10 we show

Lemma 11. *There exist $K, \epsilon > 0$ such that if $v \in V(\delta(\epsilon))$ and $\|y\| < \epsilon$, then there is a $T \in \mathbb{R}$ such that for $t \in [T, 2T]$ we have $\mathcal{P}_t(V(\delta(\epsilon))) \subset V(\delta(\epsilon))$ and, with $(0, z) = \phi^t(0, y)$,*

$$\|v(0, y)\| < K\|y\|^\alpha \to \|(\mathcal{P}_t v)(0, z)\| < K\|z\|^\alpha.$$

Inductively this yields

Corollary 12. *There exist $K, \epsilon > 0$ such that for $v \in V(\delta(\epsilon))$ and $\|y\| < \epsilon$ there is a $T \in \mathbb{R}$ such that for $t > T$ we have $\mathcal{P}_t(V(\delta(\epsilon))) \subset V(\delta(\epsilon))$ and $\|v(0, y)\| < K\|y\|^\alpha \Rightarrow \|(\mathcal{P}_t v)(0, z)\| < K\|z\|^\alpha$.*

If we take η to exceed the slowest contraction rate, then Lemma 10 follows by Lemma 8 and we are done.

Proof of Lemma 11. Write $v(y)$ instead of $v(0, y)$, etc. Then $v(y)$ is the graph of a linear map D and hence the image of the map $\begin{pmatrix} I \\ D \end{pmatrix}$ where I is the (k, k)-identity matrix. Thus

$$(D\phi^t|_y)(v(y)) = \begin{pmatrix} A_t & 0 \\ B_t & C_t \end{pmatrix}\begin{pmatrix} I \\ D \end{pmatrix} = \begin{pmatrix} A_t \\ B_t + C_t D \end{pmatrix} \sim \begin{pmatrix} I \\ (B_t + C_t D)A_t^{-1} \end{pmatrix},$$

where " $\sim$ " indicates that the two maps have the same image. If $\|v(y)\| \leq K\|y\|^\alpha$, $z = \phi^t y$, and T is such that $C^{2+\alpha}\left(\nu_s(q, t)^{-1}\mu_s(q, t)\mu_f(q, t)^{-\alpha}\right) < \frac{1}{2}$ and $\mathscr{P}_t(V(\delta)) \subset V(\delta)$ for $t > T$, then

$$\begin{aligned}
\|D(z)\| &= \|(B_t(y) + C_t(y)D(y))A_t^{-1}(y)\| \\
&\leq \|B_t(y)\|\,\|A_t^{-1}(y)\| + \|A_t^{-1}(y)\|\,\|C_t(y)\|\,\|D(y)\| \\
&\leq C_t\|y\| \cdot C\nu_s(q, t)^{-1} + C^2\nu_s(q, t)^{-1}\mu_s(q, t)K\|y\|^\alpha \\
&\leq C_t C^2 \mu_f(q, t)^{-1}\nu_s(q, t)^{-1}\|z\| \\
&\quad + C^{2+\alpha}\left(\nu_s(q, t)^{-1}\mu_s(q, t)\mu_f(q, t)^{-\alpha}\right)K\|z\|^\alpha \\
&< K\|z\|^\alpha
\end{aligned}$$

for $t \in [T, 2T]$ whenever $K > \sup_{t\in[T,2T]} 2C_t C^2\mu_f(q, t)^{-1}\nu_s(q, t)^{-1}$. q.e.d.

Theorem 5 now follows after similarly modifying [1] for $\alpha \geq 1$ and getting the same regularity for E^s by reversing time. q.e.d.

For Theorem 6 we need a lemma from ordinary differential equations.

Lemma 13. (*Gronwall's inequality*). *If* $f, g \in C^0([0, \infty), (0, \infty))$, $\alpha \in \mathbb{R}_+$, *and* $f(t) \leq \alpha + \int_0^t f(s)g(s)\,ds$, *then* $f(t) \leq \alpha e^{\int_0^t g(s)\,ds}$. *The same holds with reversed inequalities, so* $0 < h(t) \leq f'(t)/f(t) \leq g(t)$ *implies* $f(0)e^{\int_0^t h(s)\,ds} \leq f(t) \leq f(0)e^{\int_0^t g(s)\,ds}$.

Proof. Integrating $f(t)g(t)/(\alpha + \int_0^t f(s)g(s)\,ds) \leq g(t)$ yields

$$\log\left(\alpha + \int_0^t f(s)g(s)ds\right) - \log\alpha \leq \int_0^t g(s)ds;$$

hence $f(t) \leq \alpha + \int_0^t f(s)g(s)\,ds \leq \alpha e^{\int_0^t g(s)\,ds}$. Same with " $\geq$ ". q.e.d.

Proof of Theorem 6. This is an adaptation of the arguments in [6, Theorem 3.2.17]. Fix $\tau > 0$ and a continuous family of symmetric operators E from the horizontal subspace V_h in TSN to the vertical subspace $V_v \simeq V_h$. For $p \in SN$, take the geodesic with $\dot{c}(0) = p$, and let $E_\tau(p) = (g^\tau)^*(E(\dot{c}(-\tau)))$ be the image of $E(\dot{c}(-\tau))$ under the geodesic flow, whose action is given by the Riccati equation $\dot{E}(t) + E^2(t) + K(t) = 0$

along c. So if $K_1(t) := -\inf K_{c(t)}$ and $K_2(t) := -\sup K_{c(t)}$, both taken over all two-dimensional subspaces, then $\beta(t) := \min_{v \in S_pM} \langle E_t(v), v\rangle > 0$ and $\gamma(t) := \max_{v \in S_pM} \langle E_t(v), v\rangle > 0$ satisfy differential inequalities $\dot\beta \geq K_2 - \beta^2$ and $\dot\gamma \leq K_1 - \gamma^2$ along c. By relative a-pinching, $K_2 > aK_1$, so whenever $\beta(t) \leq a\gamma(t)$ we have

$$\dot\beta\gamma - \beta\dot\gamma \geq (K_2 - \beta^2)\gamma - \beta(K_1 - \gamma^2) > (a\gamma - \beta)K_1 + \gamma\beta(\gamma - \beta)$$
$$> (a\gamma - \beta)(K_1 + \gamma\beta) \geq 0$$

and

$$\frac{d}{dt}\frac{\beta}{\gamma}(t) > 0.$$

Thus $\beta > a\gamma$ for all t as long as we take $\beta(0) > a\gamma(0)$. The spectrum of $U := \lim_{\tau\to\infty} E$ is thus in $[\kappa_0(p), \kappa_1(p)]$ for Hölder continuous $\kappa_i : SN \to \mathbb{R}_+$ with $a\kappa_1 \leq \kappa_0$. (Here one would like $\sqrt{a}$ instead.)

U represents the unstable distribution in the sense that every $v \in E^u(p)$ can be written as $(v_h, U(p)v_h)$ for some horizontal vector $v_h \in T_pSN$. In effect, v_h gives the initial value of an unstable Jacobi field along c, and $Uv_h = \nabla_{\dot c(0)} v_h$ gives the initial derivative. Along c we write $\kappa_i(t)$ for $\kappa_i(\dot c(t))$ and $U(t)$ for $U_i(\dot c(t))$. Then

$$(1)\quad 2\kappa_0(t)\|v_h(t)\|^2 \leq \frac{d}{dt}\|v_h(t)\|^2 = 2\langle v_h(t), U(t)v_h(t)\rangle \leq 2\kappa_1(t)\|v_h(t)\|^2,$$

$$(2)\quad \kappa_0^2(t)\|v_h(t)\|^2 \leq \|\nabla v_h(t)\|^2 = \|U(t)v_h(t)\|^2 \leq \kappa_1^2(t)\|v_h(t)\|^2.$$

With $x_i(t) := \int_0^t \kappa_i(s)\,ds$, Lemma 13 and (1) give

$$\|v_h(0)\|^2 e^{2x_0(t)} \leq \|v_h(t)\|^2 \leq \|v_h(0)\|^2 e^{2x_1(t)},$$

which together with (2) yields

$$\frac{\kappa_0^2(t)}{\kappa_1^2(0)}\|\nabla v_h(0)\|^2 e^{2x_0(t)} \leq \kappa_0^2(t)\|v_h(0)\|^2 e^{2x_0(t)} \leq \kappa_0^2(t)\|v_h(t)\|^2 \leq \|\nabla v_h(t)\|^2$$
$$\leq \kappa_1^2(t)\|v_h(t)\|^2 \leq \kappa_1^2(t)\|v_h(0)\|^2 e^{2x_1(t)} \leq \frac{\kappa_1^2(t)}{\kappa_0^2(0)}\|\nabla v_h(0)\|^2 e^{2x_1(t)}.$$

Since the κ_i are bounded, the last two equations show that

$$\frac{1}{C}\|v(0)\|e^{x_0(t)} \leq \|v(t)\| \leq C\|v(0)\|e^{x_1(t)}.$$

So if $p = \dot c(0)$ then $\nu_s(p, t) \geq e^{x_0(t)}/C$, $\nu_f(p, t) \leq Ce^{x_1(t)}$, and

$$\nu_s(p, t)^{-2/2a}\nu_f(p, t) \leq C' e^{(1/a)\int_0^t a\kappa_1(s) - \kappa_0(s)\,ds}.$$

By compactness relative a-pinching implies relative $(a+\epsilon)$-pinching for some $\epsilon > 0$, so the integrand is bounded away from zero. This implies $2a$-bunching and also $(2a+\epsilon)$-bunching by the same token. q.e.d.

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